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Matrix Schwarzschild black holes in large N limit

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Abstract: Based on a gas picture of D0-brane partons, it is shown that the entropy, as well as the geometric size of an infinitely boosted Schwarzschild black hole, can be accounted for in matrix theory by interactions involving spins, or interactions involving more than two bodies simultaneously.

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Contents

1 Introduction

One of the more interesting applications of matrix theory [1] is to the study of quantum properties of black holes. Some first steps in this direction have been taken in [2, 3, 4]. More recently, discussions on Schwarzschild black holes at a special kinematic point are presented in [5]. However, as pointed out in [6], the work of [5] represents certain understanding of the black string side right before the black string collapses in the longitudinal direction to form a black hole.

To truly understand the hole regime in the infinite momentum frame (IMF), one has to look at the limit $N >> S$, where N is the number of partons and S is the entropy of the hole. For a given rest mass M, only in this limit could one hope that the entropy becomes independent of how much one boosts the black hole. And it appears, as also emphasized in [6], that only the zero-modes of the relevant large N Yang-Mills theory are significant in forming the cluster, as the temperature is so low in the IMF. We shall present a plausible picture for understanding Schwarzschild black holes above four dimensions along this line.

Compactifying M-theory on T^d, we are left with $D = 11 - d$ dimensional uncompactified spacetime. For simplicity, T^d is assumed to be a rectangular torus with equal circumference L. To formulate the matrix theory, we need to compactify one longitudinal dimension too with radius R, and eventually take the limit $R \to \infty$ after relevant calculations are done. We start with the standard formulas concerning a Schwarzschild black hole

$$R_s = (G_D M)^{\frac{1}{D-3}}, \quad S = R_s^{D-2}/G_D = M^{\frac{D-2}{D-3}} G_D^{\frac{1}{D-3}} , \tag{1.1}$$

where $G_D = l_p^9/L^d$ is the Newton constant in D dimensions. These two equations are the ones we wish to derive in the matrix theory context up to numerical coefficients. The on-shell relation in the IMF

$$E_{LC} = \frac{M^2 R}{N}$$

together with the thermodynamics relation $E_{LC} = ST$ results in

$$S = \frac{M^2 R}{NT} \,. \tag{1.2}$$

As long as $D > 4$, this relation and the second equation in (1.1) combine to yield

$$M = (NT/R)^{\frac{D-3}{D-4}} G_D^{\frac{1}{D-4}} \,, \tag{1.3}$$

and

$$S = (NT/R)^{\frac{D-2}{D-4}} G_D^{\frac{2}{D-4}} \,. \tag{1.4}$$

This is the equation of state predicted by the Bekenstein-Hawking entropy formula. Furthermore, we have

$$R_s = (NTG_D/R)^{\frac{1}{D-4}} \,. \tag{1.5}$$

We used the integrated relation $E = TS$ in deriving the above relations. To guarantee such a relation to yield from the first law $dE = TdS$ of thermodynamics, certain statistics of the constituents such as the Boltzmann statistics is required. We will return to this problem in sect.4.

In the next few sections we aim to give an explanation of eqs.(1.4) and (1.5).

2 The hole limit

We are interested in the hole limit where $R >> R_s$. With the help of (1.5) this is

$$R^{D-4} >> \frac{NTG_D}{R} \,. \tag{2.1}$$

For the IMF physics to work effectively, $N/R >> M$, this together with (1.3) gives

$$R >> NT^{D-3} G_D \,. \tag{2.2}$$

The two conditions (2.1) (2.2) imply $N >> RM >> R_s M = S$, that is, we are far from the point $N = S$ considered in [5] and [6]. Finally, to trust thermodynamics, $S >> 1$ which implies

$$R << NTG_D^{\frac{2}{D-2}} \,. \tag{2.3}$$

Combination of three inequalities (2.1) (2.2) (2.3) for R yields conditions on the temperature

$$NT >> G_D^{-\frac{1}{D-2}} >> T \,, \tag{2.4}$$

where $G_D^{-\frac{1}{D-2}}$ is just the Planck mass in D dimensions.

As we shall show toward the end of this paper, the temperature is so low that it is difficult, if not impossible, to do thermodynamics on the dual torus $\tilde{T}^d$ on which the

Supersymmetric Yang-Mills is defined. This indicates that, the physics is dominated by the zero-modes representing D0-brane dynamics, and the desired formulas (1.4) and (1.5) must be derived within the picture of D0-brane gas. Later we shall show that indeed the Born-Oppenheimer approximation is good in the large N limit provided $D > 4$, and this is just the condition for the validity of (1.4) and (1.5).

3 The interacting gas picture

As argued convincingly in [6], the hole limit must be dominated by the zero-mode dynamics of the underlying large N theory. Our first piece of firm evidence for this is the estimate of the geometric radius R_s, given knowledge about the entropy as in (1.4). We adapt a calculation in the first paper of [5]. Within the gas picture, the black hole is thought of as a long-lived bound state of partons, and the virial theorem is applicable here. The kinetic energy of partons and the total energy are of the same order, thus

$$Nm < v^2 >= TS = T(NT/R)^{\frac{D-2}{D-4}} G_D^{\frac{2}{D-4}} \ . \tag{3.1}$$

Now the mean velocity is determined by the size of the bound state R_s and the typical frequency which we take to be T, so $v \sim TR_s$. Substituting this into the above equation we obtain a relation between R_s and S. Indeed, given the R.H.S. of (3.1) we determine

$$R_s = (NTG_D/R)^{\frac{1}{D-4}} \ , \tag{3.2}$$

precisely the desired result (1.5). The fact that such a simple estimate gives us the correct scaling strongly suggests that the interacting gas picture is a good one. The virial theorem thus reduces the two independent unknowns R_s and S to only one. If there is a way to determine one of them, then both are determined correctly.

Our strategy for determining R_s is the following. We first postulate some relevant forms of interaction energy, and use the desired result for R_s to determine them. We then argue that these interactions exist in matrix theory, and assuming certain correlations among spin and orbital motion, these are dominant interactions, thus justifying the calculation.

For simplicity, we assume the dominant interaction depend on the mean velocity and the mean separation between partons in a power law fashion. Spins will be important, and for our purpose we can always choose a proper normalization such that they do not figure in for the moment. The total interaction will also depend on N, after summing up over partons. Let $V_l = C_{N,l} v^{l+1}/r^n$ be the total interaction energy. The meaning of l will become clear momentarily. By the virial theorem,

$$Nmv^2 \sim C_{N,l} v^{l+1}/R_s^n \ . \tag{3.3}$$

Plugging $v \sim TR_s$ into (3.3), and reading off the dependence of R_s on T, we determine n

$$n = (l-1)(D-3) \ , \tag{3.4}$$

in order to match on to (1.5). To match the whole formula, then $C_{N,l} = (N/R)^l G_D^{l-1}$, so the total interaction energy is

$$V_l = \left(\frac{N}{R}\right)^l G_D^{l-1} \frac{v^{l+1}}{r^{(l-1)(D-3)}} . \tag{3.5}$$

Now, the dependence of V_l on N suggests to us that the origin of V_l is a l-body interaction, since the total number of l-tuples is of the order N^l. The first choice is $l = 2$, the power in v goes as v^3 and implies that some dependence on spin is needed. The second choice is $l = 3$, a 3-body interaction. As we shall show, for all l, the interaction as given in (3.5) is possible in matrix theory. Before doing that, we want to compare the contribution of V_l to the well-known two-body interaction $U_2 = N^2(G_D/R^3)v^4/r^{D-4}$. Since all V_l are of the same order, it is sufficient to do this for V_2:

$$\frac{V_2}{U_2} = \frac{R}{vR_s} = \frac{N}{R_s^{D-2}/G_D} = \frac{N}{S} >> 1 . \tag{3.6}$$

where we used the formula for R_s to express T in terms of other quantities. Indeed, all the interaction forms (3.5) dominate over the standard velocity dependent two-body interaction in the regime where we have an infinitely boosted black hole.

Next we turn to the issue whether these desired interactions can actually arise in matrix theory. The l-body interaction (3.5) is to be calculated as a scattering amplitude in which l partons scatter into l partons. For a given Feynman diagram in the matrix quantum mechanics, typically one need to insert an operator for each outleg. Now there are $2l$ outlegs, and only $l+1$ velocity factors, apparently we need a factor $(\psi^2)^{l-1}$ to make up all the insertions, where ψ is the fermion of 16 components. Thus, schematically, the l-body interaction is

$$u_l = (\frac{G_D^{l-1}}{R^l})v^{l+1}(\psi^2)^{l-1}/r^{(l-1)(D-3)} . \tag{3.7}$$

Note that, for $l = 2$, this two-body spin dependent force is computed in [8] and discussed in [9], whose existence is therefore confirmed. For a recent calculation in matrix theory, see [10].

To see whether other l-body interactions can be derived in matrix theory, we first concentrate on the case $D = 11$, and then argue for general D. To this end, we need to write down the matrix action schematically

$$S = \frac{1}{\hbar}\int dt \mathrm{Tr}\left((\partial_t X)^2 + [X,X]^2 + \psi\partial_t\psi + \psi[X,\psi]\right) , \tag{3.8}$$

where $\hbar = R^3/l_p^6$. In putting the action into the form (3.8), we have rescaled $X \to (l_p^3/R)X$, so that X in (3.8) has the dimension $[X] = L^{-1}$. Similarly, $[\psi] = L^{-3/2}$. Now, $[v^{l+1}(\psi^2)^{l-1}/r^{8(l-1)}] = L^{3l-7}$, where we have put $D = 11$. To obtain a term with dimension of energy, a factor $\hbar^{l-2}$ is to be inserted. This means that the l-body effect is of $l-1$ loops. For $l = 2$, this agrees with the analysis of [8, 9]. For $l = 3$, this is a two-loop effect. Next, we want to check whether the dimensional coefficient comes out correctly. Rescaling r back, we have a factor $(l_p^3/R)^{7l-9}$, this together with $\hbar^{l-2} = (R^3/l_p^6)^{l-2}$ gives

$R^{-l}(l_p^{15}/R^3)^{l-1}$. Finally, ψ^2 as a spin factor scales as $\hbar$, so $(\psi^2)^{l-1}$ contributes a factor $\hbar^{l-1} = (R^3/l_p^6)^{l-1}$. This combined with the previous factor we obtained gives $R^{-l}l_p^{9(l-1)} = R^{-l}G_{11}^{l-1}$, the right combination appearing in (3.7).

Demanding that the above result directly generalizes to D dimensions requires the distance dependent part assume a special form. For the l-body interaction, one has to sum over periodic images of $l-1$ partons on T^d. Pick x_1 out and assume that the dependence of the separations is $\prod_{i=2}^{l}|x_1 - x_i|^{-8}$. Now summing over all the images of x_i ($i \geq 2$) one obtains $L^{-d(l-1)}\prod_{i=2}^{l}|x_1 - x_i|^{-(D-3)}$. The factor $L^{-d(l-1)}$ is precisely the one needed to yield G_D^{l-1} from G_{11}^{l-1}. For $l = 3$, we note that a similar 3-body interaction is discussed in a recent paper [11]. Our interaction $v^4(\psi^2)^2/r^{16}$ as in $D = 11$ is a superpartner of v^6/r^{14}. The term ruled out in [11] however is not the same term as ours, since the separation dependent part of that term depends on all $r_{ij} = |x_i - x_j|$, and there it is assumed $r_{12} \sim r_{13} = R >> r_{23} = r$.

Finally, we need to justify the Born-Oppenheimer approximation. Following [7], for a typical velocity v, there is a characteristic size called the stadium size, below which the the Born-Oppenheimer approximation breaks down. In D dimensions, the stadium size is just $\sqrt{v}\sqrt{l_p^3/R}$. Now $v \sim TR_s$, so the stadium size goes like $\sqrt{TR_s}\sqrt{l_p^3/R}$. For a given cut-off R (no matter how large it is) and a given fixed horizon size R_s, the temperature scales as $1/N$ according to (1.5) in the large N limit. Thus the stadium size scales as $1/\sqrt{N}$. The mean separation between the nearest two partons is $R_s/N^{1/(D-2)}$, and is much larger than $1/\sqrt{N}$ if $D > 4$. This condition is precisely the condition for both equations (1.5) and (1.4) to be valid.

4 Subtleties

There are a number of subtleties one could raise to oppose the ideas put forward here. We shall mention only a couple of them.

The first question is, if some special spin dependent interactions are important for understanding the large N limit of a black hole, what about other spin dependent interactions? We have seen that the spin orbital coupling $v^3\psi^2/r^8$ dominates the familiar force v^4/r^7 (in $D = 11$), assuming that there is a correlation between spin and orbital motion so that this force is not averaged out. This is because the typical velocity in the large N limit is very small, and the size of the black hole is fixed. Thus the smaller power in the velocity, the more important a term is. For instance, the spin dependent force $(l_p^9/R)v^2(\psi^2)^2/r^9$, if not vanishing, is larger than the one we considered. However, if we use this term as the interaction energy in (3.3) to determine R_s, we will find that $R_s = N^{1/9}l_p$. The size of the cluster blows up in the large N limit, not the canonical behavior of a boosted transverse object. If the size does not behave canonically, it is hard to demand the rest mass to behave canonically. In such a case, the mean velocity is no longer suppressed by $1/N$, and we do not know whether the v^2 interaction continues to dominate the v^3 interaction. The same can be said of other types of spin dependent interactions.

It is shown in [6] that at the transition point $N \sim S$ where a black string becomes unstable and collapses to form a black hole, one can use the two body interaction v^4/r^7 to determine the size of the black hole, and of course the rest mass does not behave canonically. It is easy to check that all super-partners of the v^4 interaction are equally important in this regime. This fact together our above discussion suggests that, in order to have the canonical large N behavior, certain spin interaction as the one proposed in the last section must be dominating, and the interacting gas must be highly coherent such that other potentially more important spin dependent interactions are actually switched off.

The second subtlety concerns the low velocity. We have assumed $v \sim TR_s$. For a fixed R_s, T is very low in the large N limit. It is easy to see that starting from a certain N ($N = S$), the mean velocity begins to become smaller than $1/R_s$ as set by the uncertainty relation. One easy, but not constructive, way to get around of this problem is to assume the strong holographic principle hold [6]. In this case, the virtual size of the cluster is not R_s, but some other scale much larger than R_s. According to this strong holographic principle, one parton must occupy at least a unit Planck cell, therefore the virtual transverse volume of the cluster must be at least proportional to N. Another, we consider more attractive resolution is to assume that the infinitely boosted black hole is not a gas of partons, but a gas of threshold bound states of certain size. Since our main equations (3.1) and (3.5) involve the parameters N and R only through the combination $p_+ = N/R$, our calculations in the last section will go through if we replace partons by threshold bound states of fixed size. It is possible to choose the mass for these bound states such that the uncertainty relation is not violated. Indeed, assuming the uncertainty relation be saturated, using $v \sim TR_s$ and the relation between T and R_s as given in (1.5), we find that the mass of the threshold bound state is $N/(SR)$. To make up the total longitudinal momentum p_+, there are precisely S such bound states in the black hole. This is an indication that in the large N limit, not all degrees of freedom, except only part of them, are necessary for accounting for the black hole entropy. One might wonder in such a case that whether it is still necessary to employ the spin dependent interactions of the last section. For instance, can one use the familiar force v^4/r^7 between two threshold bound states to obtain the desired result within the gas picture? The answer to this question is no. To see this, recall that the corresponding 11D amplitude of two supergravitons is proportional to $p_{11}(1)p_{11}(2)$ [1], so the reduced amplitude is proportional to $(1/R)p_{11}(1)p_{11}(2) = N^2/(S^2R^3)$, thus the total interaction energy would still be proportional to N^2/R^3 which does not lead to the correct answer.

The possibility that the black hole is best viewed as a composite of S clusters rather than simply N partons is supported by the fact that the Hawking radiation is dominated by the threshold bound states of longitudinal momentum $N/(SR)$. Each of the cluster does not carry exactly the longitudinal momentum $N/(SR)$. The fluctuation in longitudinal momentum enables one to distinguish them, thus the counting of states will go more like the counting of states as if the constituents are Boltzmann particles. This explains why the entropy of the black hole is order S, and the integrated version of the first law

$E = TS$ we have used so far. In reality the structure of the black hole in the large N limit is much more complicated, and the membrane like structure as proposed in [15] might exist. There at the point $N = S$ such mechanism is employed to explain the Boltzmann statistics of the constituent partons.

We do not exclude other possible resolution to the above puzzle.

5 Fractal dimensions?

It was suggested in the first paper of [5] that under the condition $N >> S$ a new phase appears in the super Yang-Mills theory (SYM) (for $d \le 3$). If one is to do thermodynamics on the dual torus $\tilde{T}^d$, one is not to expect to excite the usual momentum modes in the relevant SYM. These modes are necessarily longitudinal objects, as pointed out in [6], and therefore have nothing to do with the infinitely boosted black hole. However, there still remains the possibility of exciting some transverse modes, as the light-cone energy of these modes is typically suppressed by a factor $1/N$, which is of the same order as the temperature $T \sim 1/N$. It appears that this scenario is in contradiction with what we have suggested, that the black hole must be thought of as a highly coherent interacting gas. Actually as far as we can see, there is no direct conflict. This is because, so far all the known transverse objects are described as some global objects living on $\tilde{T}^d$, for instance a transverse membrane corresponds to a toron.

In this section we point out a curious form of formulas (1.4) and (1.5) which might lead us to some understanding in the context of SYM. Let the size of the dual torus be Σ. To see the dependence on the dual torus explicitly, we plug $L = l_p^3/(R\Sigma)$ into $G_D = l_p^9/L^d$. We now have

$$\begin{aligned} S &= N(N^{\frac{1}{d}}\Sigma)^{\frac{2d}{7-d}}(R/l_p^2)^{\frac{3(d-3)}{7-d}}T^{\frac{9-d}{7-d}}, \\ R_s &= l_p(N^{\frac{1}{d}}\Sigma)^{\frac{d}{7-d}}(R/l_p^2)^{\frac{d-1}{7-d}}T^{\frac{1}{7-d}}. \end{aligned} \tag{5.1}$$

Now, the dependence on N and Σ combines into the effective size $N^{1/d}\Sigma$, reminding us the mechanism employed by Maldacena and Susskind to explain the fat black hole in the D-brane context [12]. The physical picture for this is that all N Dd-branes on $\tilde{T}^d$ are connected to form a single large Dd-brane by switching on Wilson lines in all directions. Since Σ is enhanced, it is possible to excite soft modes whose energy scales as $1/(N^{1/d}\Sigma)$. And instead of N^2 species of light modes, there are now only N species, this explains the extra factor N in the formula for S. There is also a nice interpretation for the factor R/l_p^2. As we have seen in section 2, this is proportional to the parton coupling constant.

The above interpretation is quite appealing, nevertheless there is a big loop hole. The above argument is valid only in the case when the excitations are dominantly momentum modes, and these are not what we want to have. Even to excite momentum modes, the condition is that the temperature T must be greater than the inverse effective size. Since the temperature goes as $1/N$, this is possible only when $d = 1$. For this case, a detailed calculation shows that the condition $Sl_p > L$ must be satisfied, that is, the internal circle is not too big.

Taking equations (5.1) literally, we find that the effective dimension is not d but the fractal dimension $d_f = 2d/(7-d)$. It is smaller than d if $0 < d < 5$, and is equal to d if $d = 0, 5$. We know $d = 5$ is a special case where multiple NS5-brane theory is argued to be the correct matrix theory [13]. The strange thing happens at $d = 6$, here $d_f = 12$ and is the only case where $d_f > d$. This makes a fractal interpretation implausible. Precisely starting from T^6, it is argued [14] that there is no simple matrix formulation of M-theory. We do not know whether there is a connection between this fact and our observation.

6 Discussion and conclusion

Our discussion in this paper as well as the analysis of [6] strongly suggests that an infinitely boosted black hole can be understood on the basis of a strongly correlated gas of partons. Our treatment is universal for all dimensions, except $D = 4$ which requires a separate study. An interesting aspect of the coherent gas is that the specific heat is always negative. The main lesson learned here is that a D0-brane cluster can have vastly different behavior in different kinetic regimes. A black hole is certainly the regime where the cluster has the canonical behavior of a boosted transverse object. Another possible regime is discussed in [9], where the size of the cluster is even bigger than the one discussed in [1]. To understand more details of the working of the coherent gas for a black hole, much further work is required. In particular, if one wishes to understand the matrix Schwarzschild black hole from the standpoint of super Yang-Mills theories defined on tori, new physics is to be invoked in the large N quantum field theories. As pointed out in the previous section, the gas approach and the SYM approach do not necessarily exclude each other. It might well be that understanding gained in one approach will shed light on another approach. In any case, we expect that the black hole physics is to teach us a lot about matrix theory, if matrix theory is a viable model for M-theory.

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